MSLC Workshop Series: Math 1149 and 1150
Law of Sines & Law of Cosines Workshop

There are four tools that you have at your disposal for finding the length of each side and the measure of each angle of a triangle (i.e. solving a triangle): ***Similar Triangles, Right Triangle Trigonometry, the Law of Sines,*** **and** ***the Law of Cosines*****.** This workshop will focus on using the Law of Sines and the Law of Cosines, as well as how to identify when to use them.

1. **<u>Similar Triangles:</u>** Used when given two triangles of dissimilar size, but with identical angles. ****The use of similar triangles is not covered in Math 1149 or Math 1150.***

2. **<u>Right Triangle Trigonometry:</u>** Used when working with a triangle that has a right angle.

3. **<u>Law of Sines:</u>** Used when given an oblique (non-right) triangle containing one of the two following conditions:
   a) Two angles and one side given. Angle, Angle, Side (AAS)
   b) Two known sides as well as an angle not between the known sides. Side, Side Angle (SSA)
   - SSA can result in an ambiguous situation where two triangles will satisfy the initial condition

<u>The statements of the Law of Sines</u>

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

**Either version of the Law of Sines can be used depending on how you wish to solve the problem.*

4. **<u>Law of Cosines:</u>** Used when given an oblique (non-right) triangle containing one of the two following conditions:
   a) Three sides given. Side, Side, Side (SSS)
   b) Two known sides as well as the angle between the known sides. (SAS)

<u>The statements of the Law of Cosines</u>

$$a^2=b^2+c^2-2bc\cos A; \quad b^2=a^2+c^2-2ac\cos B; \quad c^2=a^2+b^2-2ab\cos C$$

or

$$\cos A=\frac{b^2+c^2-a^2}{2bc}; \quad \cos B=\frac{a^2+c^2-b^2}{2ac}; \quad \cos C=\frac{a^2+b^2-c^2}{2ab}$$

**Either version of the Law of Cosines can be used depending on how you wish to solve the problem.*

WHEN ATTEMPTING TO USE THE LAW OF SINES OR COSINES YOU SHOULD:

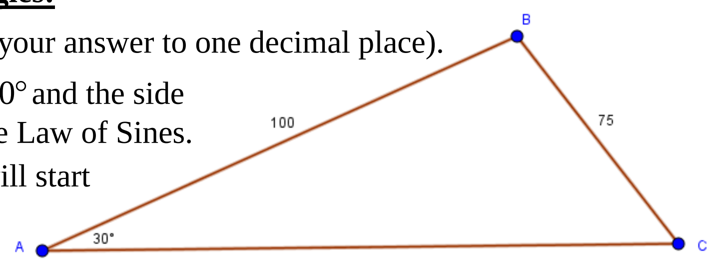
1. Check to see if you can use a less complicated technique (like right triangle trigonometry or the Pythagorean Theorem) to solve the triangle.

2. If the less complicated techniques don't apply, then determine if you need to use the Law of Sines or the Law of Cosines.
   - You will be using the Law of Sines if you are given an angle and its corresponding opposite side (for example, $\angle A$ and side a).
   - Use the Law of Cosines if you can't use the Law of Sines, that is when you are **NOT** given an angle and its corresponding opposite side.

3. Make sure your answers are reasonable.
   - The smallest side should be opposite the smallest angle, and the hypotenuse should be opposite the largest angle
   - There shouldn't be any negative sides or angles.
   - There will be at most one obtuse angle in any triangle.

Finding the remaining sides and angles of triangles:

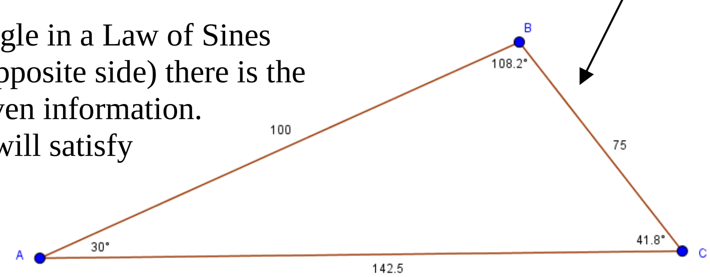
EXAMPLE 1: $\angle A = 30°$, $a = 75$, $c = 100$ (Round your answer to one decimal place).



Since we are given the measure of an angle $\angle A = 30°$ and the side opposite that angle, $a = 75$, we will need to use the Law of Sines. Because we don't know either side b or $\angle B$, we will start by finding $\angle C$.

Using the Law of Sines we get $\frac{\sin 30°}{75} = \frac{\sin C}{100}$. To get $\sin C$ by itself multiply both sides by 100 to get $\frac{100 \sin 30°}{75} = \sin C$. Now take the inverse sine of both sides and we get $C = \sin^{-1}\left(\frac{100 \sin 30°}{75}\right)$ which rounded to one decimal place tells us $\angle C \approx 41.8°$. We now add angles A and C together, then subtract that sum from $180°$ to get angle B. $(30° + 41.8°) - 180° = 108.2° = \angle B$. All that is left is for us to find side b, once again using the Law of Sines. You should use the given angle and opposite side pair, $\angle A$ and side a, to minimize the likelihood of an arithmetic error from your previous work carrying over. This gives us $\frac{75}{\sin 30°} = \frac{b}{\sin 108.2°}$. We can get b by itself by multiplying both sides by $\sin 108.2°$ to get $b = \frac{75 \sin 108.2°}{\sin 30°}$

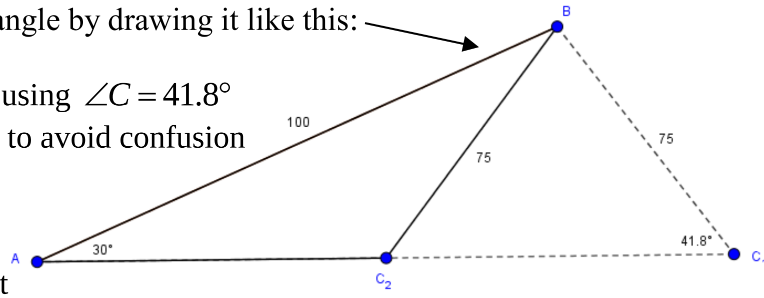
Rounded to one decimal place this gives us $b = 142.5$, meaning our triangle looks like this:



Unfortunately, when we're given two sides and an angle in a Law of Sines problem (identified by being given an angle and its opposite side) there is the possibility of two different triangles satisfying the given information. We now have to see if there is a second triangle that will satisfy our given information.

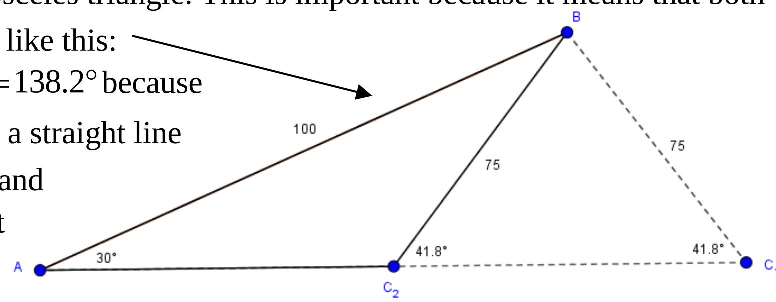
Finding the remaining sides and angles of triangles (continued):

The work we just did for the first triangle will help us determine if there's a second triangle, and if there is, what that triangle looks like. In these cases, if there is a second triangle, it will be contained inside of the first. We'll start on the second possible triangle by drawing it like this:



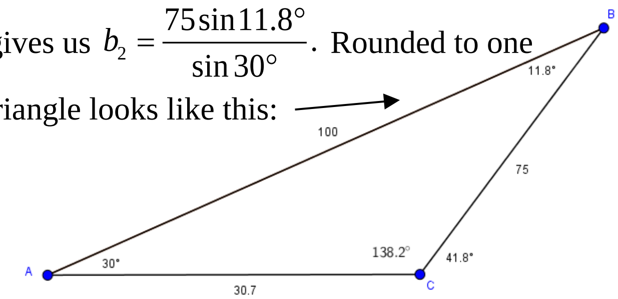
We're going to start on the second triangle by using $\angle C = 41.8°$ from our first triangle which we'll now call C_1 to avoid confusion

Side c_1 from the original triangle (dotted) and the new side c_2 (solid) are both of length 75. Those two sides together with the line segment connecting c_1 and c_2 (also dotted) form an isosceles triangle. This is important because it means that both base angles are $41.8°$. Our picture now looks like this:

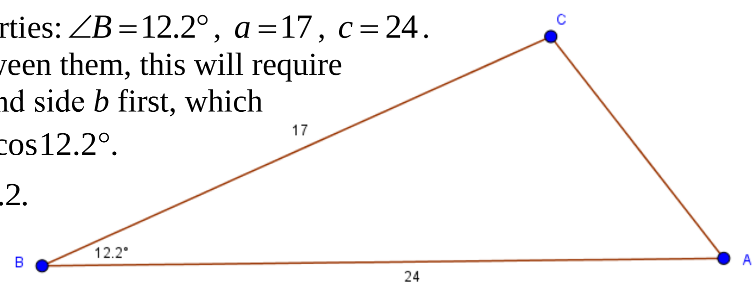


This means that in our second triangle, $\angle C_2 = 138.2°$ because The two angles that go through point c_2 form a straight line and will need to add to $180°$. Since $\angle A = 30°$ and $\angle C_2 = 138.2°$, we know $\angle B = 11.8°$. This just leaves us to find the new side b, which we're going to call b_2. We'll now use the Law of Sines one last time to find b_2.

This gives us $\dfrac{75}{\sin 30°} = \dfrac{b_2}{\sin 11.8°}$, and getting b_2 by itself gives us $b_2 = \dfrac{75 \sin 11.8°}{\sin 30°}$. Rounded to one decimal place we get $b_2 \approx 30.7$ which means our second triangle looks like this:



EXAMPLE 2: Solve the triangle with the properties: $\angle B = 12.2°$, $a = 17$, $c = 24$.

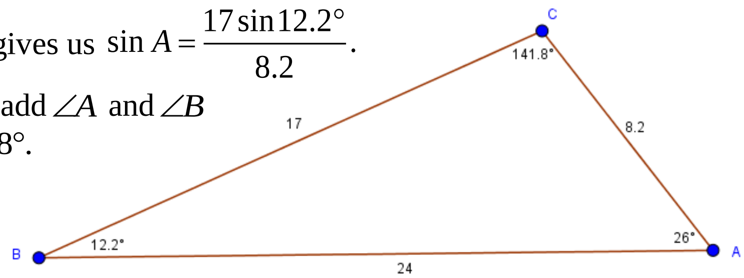


Since we are given two sides and the angle between them, this will require us to use the Law of Cosines. We're going to find side b first, which will look like this: $b^2 = 17^2 + 24^2 - 2(17)(24)\cos 12.2°$.

Rounding b to one decimal place gives us $b \approx 8.2$.
By finding b first, we can now use the Law of Sines to find one of the two unknown angles.
In these types of problems, you should find the smallest remaining angle first. (*If one of the remaining angles is obtuse, the Law of Sines will give an incorrect answer.*) We can identify the smallest remaining angle by choosing the angle that is opposite from the smallest remaining side, in this case that will be $\angle A$.

The Law of Sines yields $\dfrac{\sin 12.2°}{8.2} = \dfrac{\sin A}{17}$, which gives us $\sin A = \dfrac{17 \sin 12.2°}{8.2}$.

This tells us $\angle A \approx 26°$. To find $\angle C$ we just need to add $\angle A$ and $\angle B$ then subtract that sum from $180°$. Thus $\angle C \approx 141.8°$.



Your turn. Find the remaining sides and angles of the following triangles:
(R*ound your answers to the nearest tenth where necessary*.)

1) $\angle B = 10°$, $\angle C = 100°$, $c = 115$

2) $a = 122.5$, $b = 60.1$, $c = 154.6$

Your turn. Find the remaining sides and angles of the following triangles:
(R*ound your answers to the nearest tenth where necessary*.)

3) $\angle B = 58.3°$, $b = 13.6$, $c = 11.8$

4) $\angle A = 67.8°$, $b = 27$, $c = 26$

Your turn. Find the remaining sides and angles of the following triangles:
(R*ound your answers to the nearest tenth where necessary*.)

5) $\angle A = 45°$, $b = 18$, $a = 13$

6) $a = 500$, $b = 400$, $c = 200$