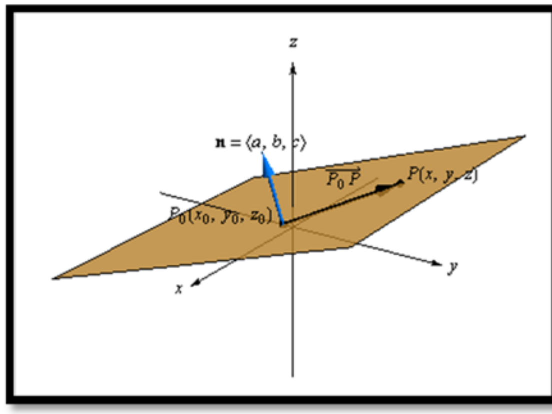
MSLC Workshop Series
Calculus II/Calculus III
Planes and Surfaces

I. Planes

A plane is like an infinite piece of paper in 3-space.

A plane can be determined if you know a **point** on the plane and a **normal vector** to the plane.

Compare this to finding the equation of a line in 2-space. You need a point to tell you the "height" and a slope or normal vector to tell you the "slant".



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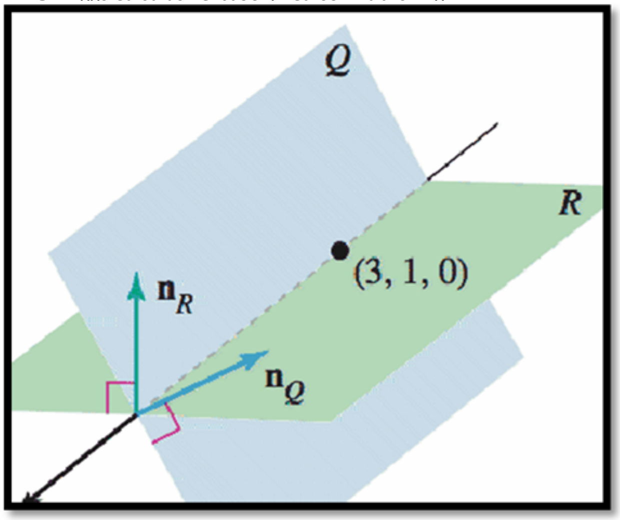
Equation of a Plane in $\mathbb{R}^3$
The plane with normal vector $\mathbf{n} = \langle a,b,c \rangle$ passing through the point (x_0, y_0, z_0) is given by the equation:
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$
Note: This can also be written as $ax + by + cz = d$.

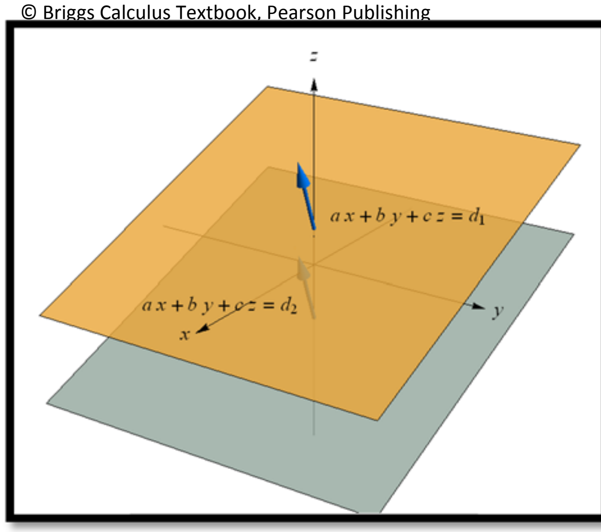
Example 1: Find the equation of the plane that contains the point (7,8,9) and is perpendicular to the vector <1,2,3>.

Example 2: Find the plane that contains the points (3,2,1), (4,5,6), and (7,8,9).

II. Intersection of Planes:



When two distinct planes intersect, they intersect in a line.



Sometimes two planes are parallel and they do not intersect.

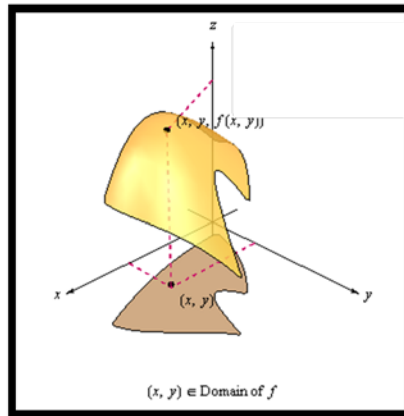
Example 1: Determine if the following planes intersect, and if they intersect, determine the line of intersection.

$$3x + 2y - 5z = 9, \quad 2x - 3y + z = 2$$

Example 2: Determine if the following planes intersect, and if they intersect, determine the line of intersection.

$$3x + 2y - 5z = 9, \quad 6x + 4y - 10z = 2$$

III. Surfaces



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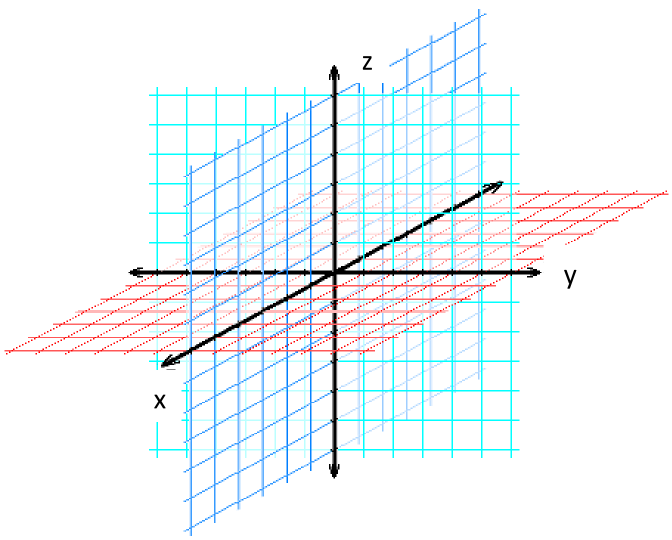
Surfaces are 2-dimensional objects. Anything that looks "locally" like a plane is called a surface. This means, if you are a tiny bug living on this surface, you think it's a plane. For example, the surface of the earth is a surface. We think it looks flat, but we know it's really the surface of a sphere.

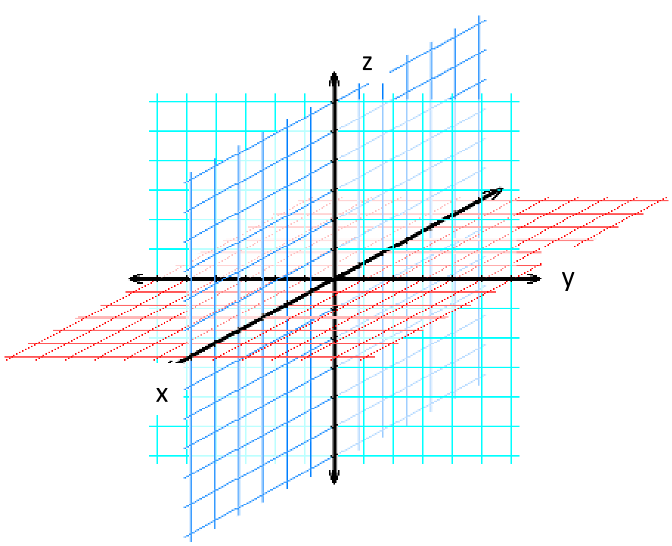
Some surfaces are graphs of equations in two variables which look like $z = f(x, y)$. We call these **functions of two variables** if they pass the vertical (i.e. parallel to z) line test.

Surfaces can be very difficult to draw or even visualize. Some are very easy, though.

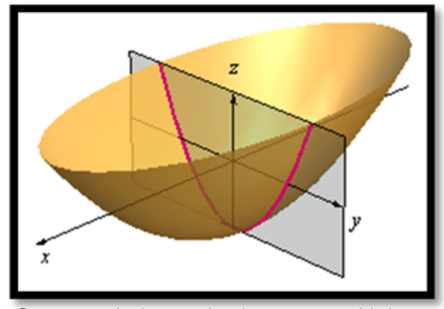
Example 1: Sketch z = 6. This is a plane.



Example 2: Sketch $z = y^2$. This is a cylinder.



V: Traces



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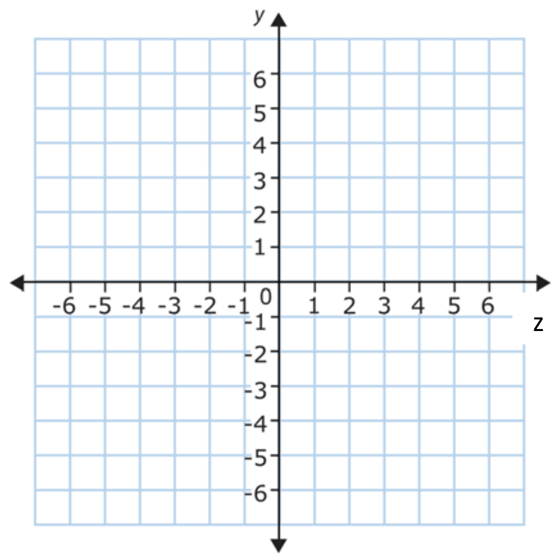
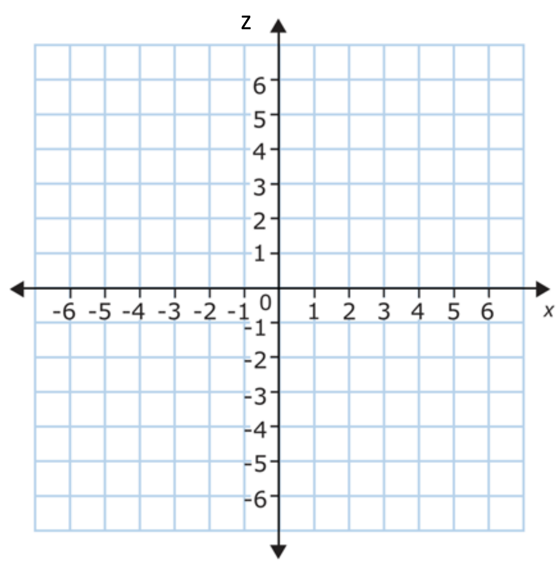
To find out more about unknown surfaces, we use various techniques to try to look at portions of these surfaces in 2-D. One such technique is called **traces**. For this technique, you intersect the surface with a plane, usually a x=a, y=b, or z=c, and look at the resulting 2-D curve.

Example 1a: Find the x=0 trace for $y = \frac{z^2}{4} + \frac{x^2}{4}$. Sketch this curve.

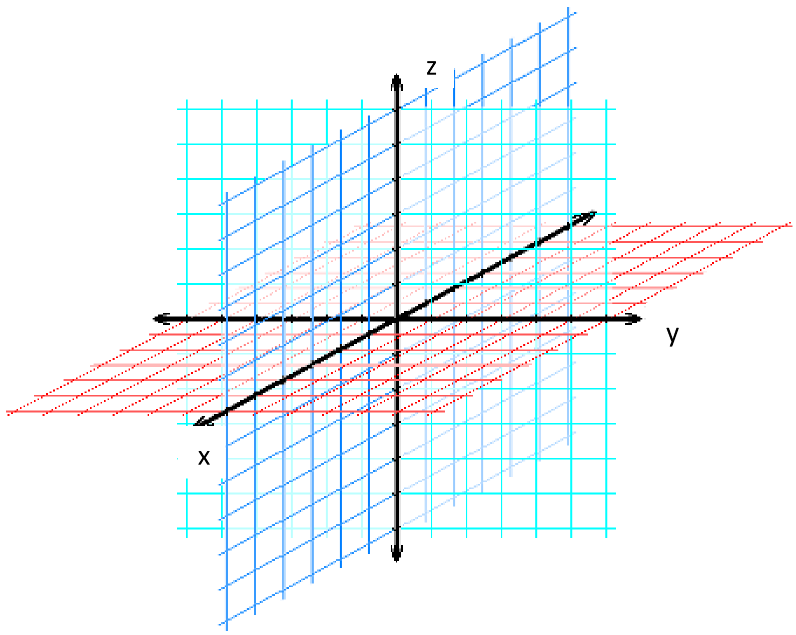


Example 1b: Find the trace when y = 4 for $y = \frac{z^2}{4} + \frac{x^2}{4}$

Sketch this curve, and compare this to your sketch on the previous page.



Example 1c: Sketch $y = \frac{z^2}{4} + \frac{x^2}{4}$



IV: Quadric Surfaces

There are some surfaces that are used so often that you really just need to memorize them. **See the chart of Quadric Surfaces on the next page** and then try the example below.

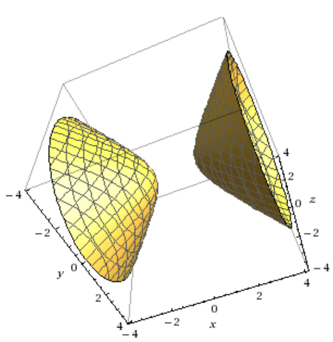
Example: Match the functions with their graphs and names.

a) $z^2 = 3x^2 + 2y^2$

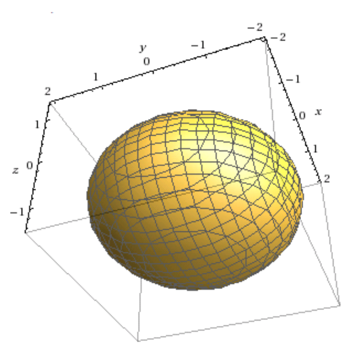
b) $x^2 - y^2 - z^2 = 1$

c) $x^2 + y^2 + 2z^2 = 4$

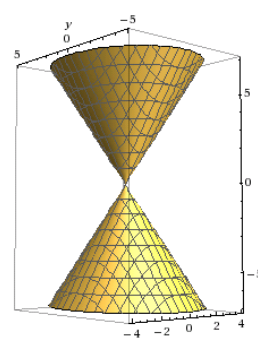
I)



II)



III)

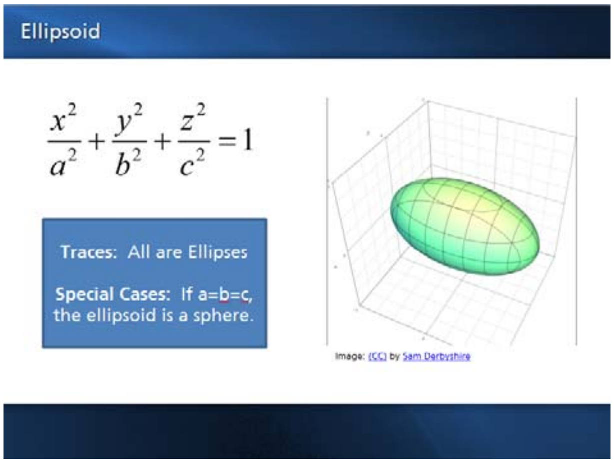


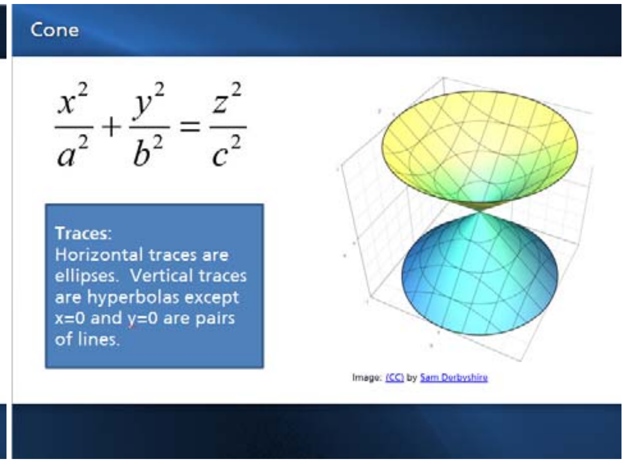
1) Hyperboloid of 2 Sheets

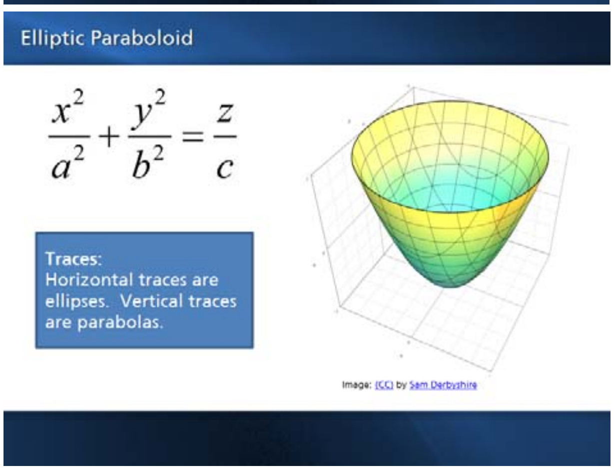
2) Cone

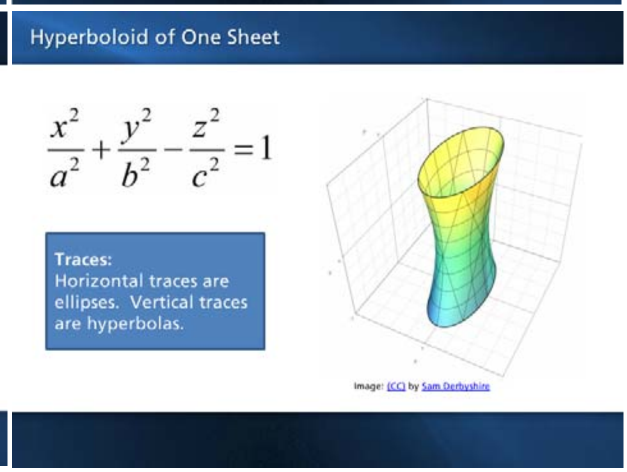
3) Ellipsoid

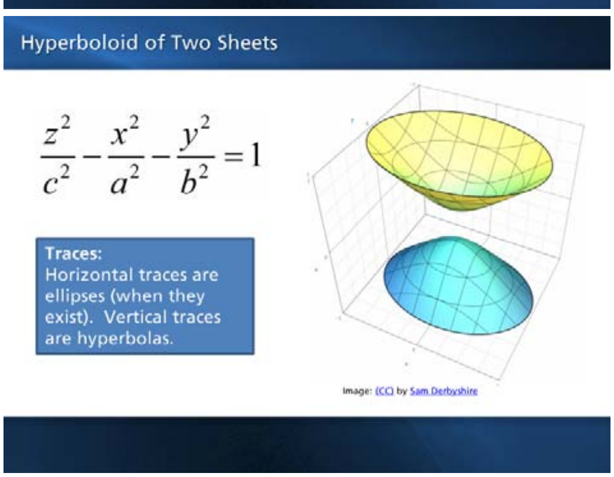
Quadric Surfaces:

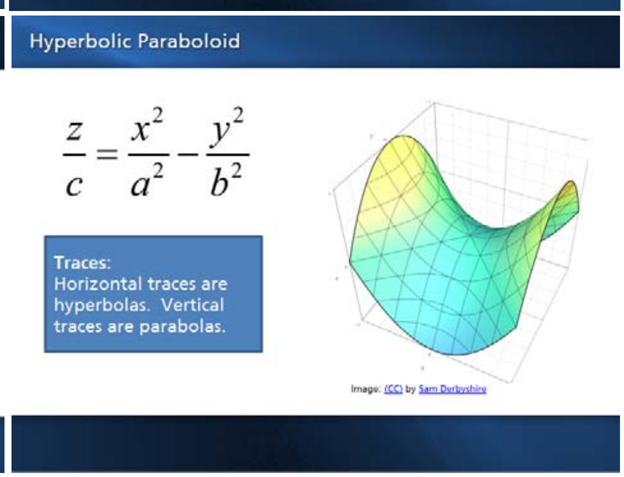












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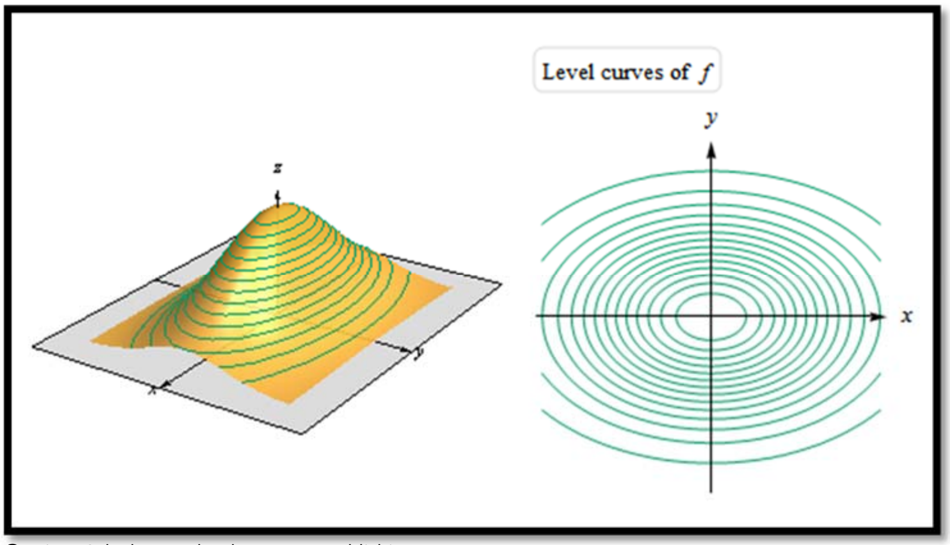
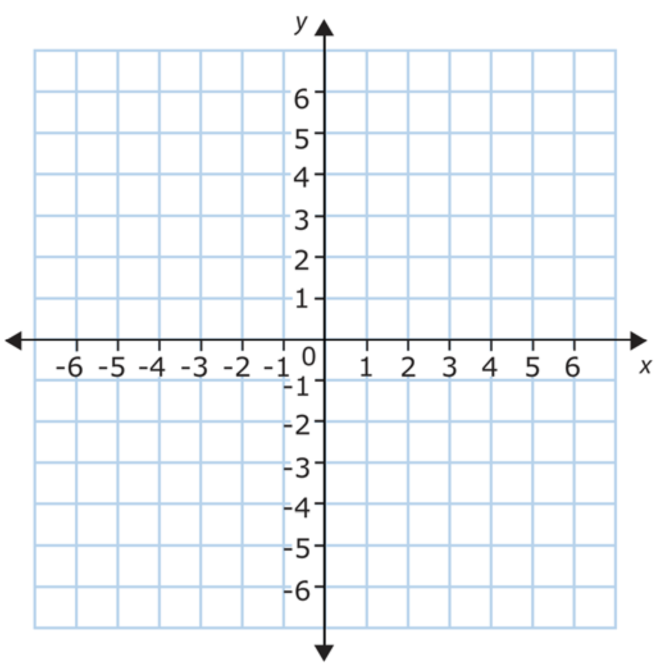
VI: Level Curves

Another technique for determining how a surface looks or behaves is by using level curves. This is used is used when the surface is a function of two variables, z = f(x,y).

Level curves are essentially a series of traces made with planes parallel to the xy-plane (z = c). These traces are all graphed on the same set of xy-axis and are used to visualize how the function changes as we change the height, z. A common example of level curves in use are elevation maps.



Example 1: Draw the level curves for $f(x,y) = y - x^2 - 1$ for f(x,y) = -2, 0, and 2.



Example 2: Draw the level curves for $f(x,y) = e^{-x^2-y^2}$ for f(x,y) = 0.1, 0.3, and 0.7.