First Fundamental Theorem of Calculus

- \( \int_a^b f(t)\,dt \) calculates the signed area under the curve \( y = f(t) \) from \( a \) to \( b \).
- Similarly, we can find the area under the curve between \( a \) and \( x \) (where \( x \) is a variable) with the integral \( \int_a^x f(t)\,dt \).
- This integral can be thought of as a function of \( x \)! Let’s call it:

\[
A(x) = \int_a^x f(t)\,dt
\]

(This function is called the accumulation function of the original function, \( f(x) \).)

Assume \( a = 0 \) and \( f(x) \) is the function below.

1. Find \( A(2) \).
2. Find \( A(9) \).
3. Is \( A \) increasing or decreasing on the interval \([5,6] \)?

The rate of accumulation at \( t = x \) is equal to the value of the function being accumulated at \( t = x \). This relationship is known as the First Fundamental Theorem of Calculus. That is,

\[
A'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x), \text{ where } f(t) \text{ is a continuous function on } [a, x].
\]
\[ A'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x), \text{ where } f(t) \text{ is a continuous function on } [a, x]. \]

**Examples:**

Find \( F'(x) \).

1. \( F(x) = \int_2^x (t^2 - \cos t + 3) \, dt \)

2. \( F(x) = \int_0^{x^2} t \cos(2t) \, dt \)

3. \( F(x) = \int_x^{x^2} \frac{3}{t} \, dt \)

4. \( F(x) = \int_x^{x^3} \sin^2(t) \, dt \)
Second Fundamental Theorem of Calculus

What if the function we are integrating doesn’t lend itself to nice geometry calculation?

- We could use Riemann Sums
- OR -
- We could use the Second Fundamental Theorem, which states

\[ \int_a^b f(x) \, dx = F(b) - F(a), \]

where \( F(x) \) is any antiderivative of \( f(x) \).

(This is much more efficient than using Riemann sums, assuming we can find an antiderivative of the function.)

**Examples:**

1. \[ \int_1^3 (2x^2 + x) \, dx \]

2. \[ \int_0^\pi \sec^2(x) \, dx \]
Substitution – Undoing the Chain Rule

Suppose we want to find $\int \sin(x^4) \, 4x^3 \, dx$. We know how to find $\int \sin(u) \, du$, so we are going to do a **CHANGE OF VARIABLE**.

\[ u = x^4 \]

But to do this, we also need to change the $dx$ into a $du$.

\[ \frac{du}{dx} = \]

Which gives us a new integral that we know how to solve:

**CONVERTING ALL $x$’s to $u$’s**

Most importantly, this change of variable will only work if we change EVERY $x$ so that the ENTIRE integral is in terms of $u$.

**When Substitution Fails:**

$\int \sin(x^4) \, dx$

**Back Substitution:**

$\int \frac{x}{x+1} \, dx$

**Limits of Integration:**

$\int_1^2 (x + 3)^6 \, dx$
Practice Problems:

1. \[ \int_{\pi/4}^{\pi} \sin x \cos x \, dx \]

2. \[ \int_{0}^{4} \sqrt{2t + 1} \, dt \]

3. \[ \int_{0}^{4} x^2 \cos(x^3 - 1) \, dx \]

4. \[ \int_{-4}^{4} \sqrt{16 - x^2} \, dx \] (Hint: remember integral means area)
5. \( \int_{-4}^{4} \frac{2}{1+4x^2} \, dx \)

6. \( \int_{-4}^{2} \sin(x + 1) \, dx \)

7. \( \int_{0}^{1} 5x^2 e^{x^3} \, dx \)

8. \( \int \frac{2x+1}{x^2+4} \, dx \)