More on Area

Recall that the definite integral \( \int_a^b f(x)\,dx \) gives us the \textit{signed} area of the region trapped between the curve \( y = f(x) \) and the \( x \)-axis on the interval \([a, b]\). This means that a positive sign is attached to areas above the \( x \)-axis, and a negative sign is attached to areas below the \( x \)-axis.

Let \( f(x) \) be the function whose graph is shown above.

Using what you know from geometry, find \( \int_0^9 f(x)\,dx \).
First Fundamental Theorem of Calculus

- \( \int_a^b f(t)\,dt \) calculates the signed area under the curve \( y = f(t) \) from \( a \) to \( b \).

- Similarly, we can find the area under the curve between \( a \) and \( x \) (where \( x \) is a variable) with the integral \( \int_a^x f(t)\,dt \).

- This integral can be thought of as a function of \( x \)! Let's call it:

\[
F(x) = \int_a^x f(t)\,dt
\]

(This function is called the accumulation function of the original function, \( f(x) \).)

Let’s try to get an idea of what this function looks like in a specific case:
(See Demo: [http://math.furman.edu/~dcs/java/ftc.html](http://math.furman.edu/~dcs/java/ftc.html))

The rate of accumulation at \( t = x \) is equal to the value of the function being accumulated at \( t = x \). This relationship is known as the First Fundamental Theorem of Calculus. That is:

\[
F'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x), \text{ where } f(t) \text{ is a continuous function on } [a, x].
\]
\[ F'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x), \] where \( f(t) \) is a continuous function on \([a, x] \).

**Examples:** Find \( F'(x) \).

1. \( F(x) = \int_2^x (t^2 - \cos t + 3) \, dt \)

2. \( F(x) = \int_x^{4.3} \frac{1}{t} \, dt \)

3. \( F(x) = \int_0^{x^2} t \cos(2t) \, dt \)

4. \( F(x) = \int_x^{x^3} \sin^2(t) \, dt \)
Second Fundamental Theorem of Calculus

What if the function we are integrating doesn’t lend itself to nice geometry calculation?

- We could use Riemann Sums
- OR -
- We could use the Second Fundamental Theorem, which states

\[
\int_a^b f(x) \, dx = F(b) - F(a), \text{ where } F(x) \text{ is any antiderivative of } f(x).
\]

(This is much more efficient than using Riemann sums, assuming we can find an antiderivative of the function.)

Examples:

1. \( \int_1^3 (2x^2 + x) \, dx \)

2. \( \int_0^{\pi/2} \sec^2(x) \, dx \)

3. \( \int_0^1 (x - 2)(3 - x) \, dx \)

4. \( \int_1^2 \frac{2x - \sqrt{x}}{x^2} \, dx \)
U-Substitution – Undoing the Chain Rule

Suppose we want to find $\int \sin(x^4) \, 4x^3 \, dx$. We know how to find $\int \sin(u) \, du$, so we are going to do a **CHANGE OF VARIABLE**.

$$u = x^4$$

But to do this, we also need to change the $dx$ into a $du$.

$$\frac{du}{dx} =$$

Which gives us a new integral that we know how to solve:

**CONVERTING ALL $x$’s to $u$’s**

Most importantly, this change of variable will only work if we change EVERY $x$ so that the ENTIRE integral is in terms of $u$.

When U-Substitution Fails:

$$\int \sin(x^4) \, dx$$

Making it match:

$$\int x^3 \sin(x^4) \, dx$$

Back Substitution:

$$\int \frac{x}{x + 1} \, dx$$

Limits of Integration:

$$\int_{1}^{2} (x + 3)^{60} \, dx$$