Sigma Notation:

Notation and Interpretation of \( \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \)

- \( \Sigma \) (capital Greek sigma, corresponds to the letter \( S \)) indicates that we are to sum numbers of the form indicated by the general term.
- \( a_k \) is the general term, which determines what is being summed, and can be defined however we want but is usually a formula containing the index: \( a_k = f(k) \).
- \( k \) is called the index; we may use any letter for the index, typically we use \( i, j, k, l, m, \) and \( n \) as indices.
- The index runs through the positive integers, starting with the number below the \( \Sigma \) (in this case 1) and ending with the integer above the \( \Sigma \) (in this case \( n \)).
- The sum on the right-hand side is the expanded form. (The \( \cdots \) contains all the terms I was too lazy to write.)
- The letter below the sigma is the variable with respect to the sum. All other letters are constants with respect to the sum.

Examples:
1. \( \sum_{k=1}^{7} k = \)
2. \( \sum_{k=3}^{9} \frac{1}{k+2} = \)
3. \( \sum_{k=1}^{9} 4 = \)

Special Sum Formulas

\[
\begin{align*}
\sum_{i=1}^{n} 1 &= n \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \left[ \frac{n(n+1)}{2} \right]^2
\end{align*}
\]

Examples:
1. \( \sum_{k=1}^{200} k = \)
2. \( \sum_{i=1}^{30} i^2 = \)

Properties of Sigma Algebra \( \Sigma \) is an operator that represents summation, and its properties are similar to the properties of addition (note what properties are not mentioned here)

- Multiplication by a common constant (also called a scalar multiple) \( \sum ca_k = c \sum a_k \)
- Addition or Subtraction (this is also called the linearity property) \( \sum (a_k \pm b_k) = \sum a_k \pm \sum b_k \)

Example:
\( \sum_{k=1}^{10} (2k^2 + k + 1) = \)
Riemann Sums: $$R = \sum_k \text{(height of } k\text{th rectangle}) \cdot \text{(width of } k\text{th rectangle})$$

**Definition of a Riemann Sum:**
Consider a function $$f(x)$$ defined on a closed interval $$[a, b]$$, partitioned into $$n$$ subintervals of equal width by means of grid points $$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$. On each subinterval $$[x_{k-1}, x_k]$$, pick a sample point $$x_k^*$$. Then the Riemann sum for $$f$$ corresponding to this partition is given by:

$$R = \sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

- **WIDTH:** $$\Delta x$$
  Since we partition the interval into evenly spaced partitions, we can calculate the width:

  $$\Delta x = \frac{b-a}{n}, \text{ where } n \text{ is the number of partitions.}$$

- **HEIGHT:** $$f(x_k^*)$$
  Also, we usually don’t pick $$x_k^*$$ arbitrarily. We use a rule to pick $$x_k^*$$. The most common rules to use are the Right Endpoint Rule, the Left Endpoint Rule, and the Midpoint Rule.

  The most common rules to use are:
  - Right Endpoint Rule: $$x_k^* =$$
  - Left Endpoint Rule: $$x_k^* =$$
  - Midpoint Rule: $$x_k^* =$$

- If we partition the interval into more and more rectangles with smaller and smaller widths, we get closer to the (signed) area trapped between the curve $$y = f(x)$$ and the $$x$$-axis.

This is where **Sigma Notation** comes in because it becomes time consuming to add up all the terms when there are many, many rectangles.
Calculating A Riemann Sum

Using the **Right Endpoint Rule**, the Riemann sum becomes:

\[\sum_{k=1}^{n} f(a + k\Delta x) (\Delta x) = \sum_{k=1}^{n} \left(\frac{b-a}{n}\right)f(a + k \frac{(b-a)}{n})\]

Using the **Left Endpoint Rule**, the Riemann sum becomes:

\[\sum_{k=1}^{n} f(a + (k - 1)\Delta x) (\Delta x) = \sum_{k=1}^{n} \left(\frac{b-a}{n}\right)f(a + (k - 1) \frac{(b-a)}{n})\]

Using the **Midpoint Rule**, the Riemann sum becomes:

\[\sum_{k=1}^{n} f\left(a + \left(\frac{(k - 1) + k}{2}\right)\Delta x\right) (\Delta x) = \sum_{k=1}^{n} \left(\frac{b-a}{n}\right)f\left(a + \left(\frac{(k - 1) + k}{2}\right) \frac{(b-a)}{n}\right)\]

**Example:**
Estimate the area under \(f(x) = x^2 + 2\) on the interval \([-2, 3]\) using right Riemann Sums and 5 rectangles.

![Graph of \(f(x) = x^2 + 2\) on interval \([-2, 3]\)]

**NOVICE (before Calculus):**
Example:
Estimate the area under $f(x) = x^2 + 2$ on the interval $[-2, 3]$ using right Riemann Sums and 5 rectangles.

SEMI-PRO (beginning Calculus student):

Example:
Estimate the area under $f(x) = x^2 + 2$ on the interval $[-2, 3]$ using right Riemann Sums and 5 rectangles.
PRO (by test time):
**Left Sum:** Estimate the area under \( f(x) = x^2 + 2 \) on the interval \([-2, 3]\) using left Riemann Sums and 5 rectangles.

**Midpoint Sum:** Estimate the area under \( f(x) = x^2 + 2 \) on the interval \([-2, 3]\) using midpoint Riemann Sums and 5 rectangles.
**Being More Accurate:**

What if we want to get a better approximation than any of the above give us? More rectangles will give us less extra or unused area between the curve and the $x$-axis. Let’s do a right sum for $\int_{-2}^{3} (x^2 + 2) \, dx$ using 5000 equal subintervals.

First calculate the width: $\Delta x = \frac{3 - (-2)}{5000}$

Then the $x$-value for the right endpoint of the $k$th rectangle is:

Thus the height of the $k$th rectangle is:

So the Riemann sum is

Now evaluate this sum using your knowledge of sigma algebra!