MSLC Math 1149 & 1150 Workshop:

Trigonometric Identities

For most of the problems in this workshop we will be using the trigonometric ratio identities below:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

If you aren't going to be given all of the Pythagorean Identities in your Trigonometry class, you don't have to worry about memorizing all of them. By using the ratio identities, the Pythagorean Identity and a little algebra you can derive the other two Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2\theta = \sec^2\theta \quad \text{and} \quad 1 + \cot^2\theta = \csc^2\theta$$

<u>Guidelines for verifying a Trigonometric Identity:</u>

1. Check whether the statement is false.
   - This is easily done on a graphing calculator. Graph both sides of the identity and check to see if you get the same picture.
2. Only manipulate one side of the proposed identity until it becomes the other side of the identity.
   - Typically the more complicated side is the best place to start. That side will give you more to work with.
3. **<u>DO NOT</u>** treat the identity like an equation.
   - This assumes that the identity is true, which is the thing that you are trying to prove.

Here are four common tricks that are used to verify an identity.

1. It is often helpful to rewrite things in terms of sine and cosine.
   a. Use the ratio identities to do this where appropriate.
2. Manipulate the Pythagorean Identities.
   a. For example, since $\sin^2 x + \cos^2 x = 1$ then $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$.
3. Use algebraic manipulations.
   a. Factor
   b. Find a common denominator
   c. Multiply the numerator and denominator by a conjugate
4. Use an additional trigonometric formula.
   a. Sum or difference formula
   b. Double-angle formula
   c. Half-angle formula

Here are five examples of verifying an identity that were worked out using these four tricks.

Verify the identity: $\tan\theta + \cot\theta = \sec\theta\csc\theta$

$$\tan\theta + \cot\theta = \sec\theta\csc\theta$$
$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \sec\theta\csc\theta$$
$$\frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta} = \sec\theta\csc\theta$$
$$\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \sec\theta\csc\theta$$
$$\frac{1}{\cos\theta\sin\theta} = \sec\theta\csc\theta$$
$$\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta} = \sec\theta\csc\theta$$
$$\sec\theta\csc\theta = \sec\theta\csc\theta$$

Verify the identity: $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$

$$(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$$
$$\csc^2\theta + \csc\theta\cot\theta - \csc\theta\cot\theta - \cot^2\theta = 1$$
$$\csc^2\theta - \cot^2\theta = 1$$
$$1 = 1$$

Verify the identity: $\frac{\sec\theta - \cos\theta}{\sec\theta + \cos\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}$

$$\frac{\sec\theta - \cos\theta}{\sec\theta + \cos\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}$$
$$\frac{\frac{1}{\cos\theta} - \cos\theta}{\frac{1}{\cos\theta} + \cos\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}$$
$$\frac{\cos\theta}{\cos\theta}\left(\frac{\frac{1}{\cos\theta} - \cos\theta}{\frac{1}{\cos\theta} + \cos\theta}\right) = \frac{\sin^2\theta}{1+\cos^2\theta}$$
$$\frac{1-\cos^2\theta}{1+\cos^2\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}$$
$$\frac{\sin^2\theta}{1+\cos^2\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}$$

Verify the identity: $(\sec x - \tan x)^2 = \frac{1-\sin x}{1+\sin x}$

$$(\sec x - \tan x)^2 = \frac{1-\sin x}{1+\sin x}$$
$$(\sec x - \tan x)^2 = \left(\frac{1-\sin x}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right)$$
$$(\sec x - \tan x)^2 = \frac{(1-\sin x)^2}{1-\sin^2 x}$$
$$(\sec x - \tan x)^2 = \frac{(1-\sin x)^2}{\cos^2 x}$$
$$(\sec x - \tan x)^2 = \left(\frac{1-\sin x}{\cos x}\right)^2$$
$$(\sec x - \tan x)^2 = \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2$$
$$(\sec x - \tan x)^2 = (\sec x - \tan x)^2$$

Verify the identity: $\cos(x+y) + \cos(x-y) = 2\cos x\cos y$

$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$$
$$(\cos x\cos y - \sin x\sin y) + (\cos x\cos y + \sin x\sin y) = 2\cos x\cos y$$
$$\cos x\cos y - \sin x\sin y + \cos x\cos y + \sin x\sin y = 2\cos x\cos y$$
$$\cos x\cos y + \cos x\cos y = 2\cos x\cos y$$
$$2\cos x\cos y = 2\cos x\cos y$$

Your turn. Verify the following trigonometric identities.

1. $\sin\theta\,(\cot\theta + \tan\theta) = \sec\theta$

2. $\dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1+\cos\alpha}$

Your turn. Verify the following trigonometric identities. (continued)

3. $2\cos x \sin y = \sin(x+y) - \sin(x-y)$

4. $1 + \sin(2\theta) = (\sin\theta + \cos\theta)^2$

Your turn. Verify the following trigonometric identities. (continued)

5. $\frac{1+\sin x}{1-\sin x} - \frac{1-\sin x}{1+\sin x} = 4 \tan x \sec x$

Summary of the rules for verifying a trigonometric identity

1. Check whether the statement is false.

2. Manipulate one side to become the other side.

3. Do not treat the identity as an equation.

If you get stuck try using a different strategy. There are several different ways to verify an identity.

Summary of the rules for verifying a trigonometric identity

1. It is often helpful to rewrite things in terms of sine and cosine.

2. Manipulate the Pythagorean Identities.

3. Use algebraic manipulations.

4. Use an additional trigonometric formula.

Here are some final advice

- There is no sure-fire way of identifying which side of an identity you should start manipulating. Practice verifying different trigonometric identities will help you identify which side works best with how you work.

- There is no nice way to tell in advance what tricks you should use, or how many steps will be necessary to verify a given identity. Practice verifying different trigonometric identities will help you identify which tricks work best in different situations.

- While you practice verifying different trigonometric identities, write down what trick you are using at each step in verifying an identity. Getting in the habit of thinking about what you want to do, and why, at each step will help you determine if you are making progress in verifying an identity or just getting more lost.