I. Parametric Curves in 2-D:

Parametric curves are curves given by \( x = g(t), \ y = h(t) \) for some independent variable \( t \), usually thought of as time. So all the points on the curve can be given by \( (x, y) = (g(t), h(t)) \).

**Caution:** These curves need not be graphs of functions.

Here are some parametric curves you should be able to recognize:

- **A line segment from \((x_1, y_1)\) to \((x_2, y_2)\):**
  
  \[
  x = x_1 + (x_2 - x_1)t \quad \text{and} \quad y = y_1 + (y_2 - y_1)t, \quad 0 \leq t \leq 1
  \]

- **A circle centered at \((x_0, y_0)\) with a radius \(a\):**
  
  \[
  x = x_0 + a \cos(bt), \quad y = y_0 + a \sin(bt)
  \]

  The circle is generated clockwise if \(b > 0\) and counterclockwise if \(b < 0\).

- **An ellipse centered at \((x_0, y_0)\):**
  
  \[
  x = x_0 + a \cos(bt), \quad y = y_0 + c \sin(bt)
  \]

  The circle is generated clockwise if \(b > 0\) and counterclockwise if \(b < 0\).

It is sometimes possible to eliminate the parameter by solving one equation for \(t\) and plugging it into the other equation. This will give the same curve, but you will lose the information about the direction and speed given by the parameter \(t\).

You can turn functions \(y = f(x)\) into parametric curves simply by letting \(x = t, \ y = f(t)\).

II. Parametric Curves in 3-D:

This is essentially exactly the same as 2-D but you get a third equation:

\[
 x = f(t), \quad y = g(t), \quad z = h(t)
\]

which gives you a point in 3-space \((x, y, z) = (f(t), g(t), h(t))\).

Eliminating the parameter of a 3-D curve will not give you a nice, single equation like in 2-D. For many purposes, parametric descriptions are the most natural way to describe higher dimensional curves.
Example: What does the following parametric equation look like? Describe its properties.

\[ x = 3 + 2\cos t, \quad y = 4 + 2\sin t, \quad z = 2t \quad \text{for} \quad 0 \leq t \leq 6\pi \]

III. Vectors:

In 3-D, it is often more helpful to talk about vectors instead of points. A vector is an object with a magnitude (length) and a direction. We draw it as an arrow.

It does not matter where a vector is sitting in space, but if a vector \((x, y, z)\) has its tail at the origin, then its head will be at the point \((x, y, z)\). (People often interchange these two related but distinct concepts.)
Vector Operations Examples:

(Table on last page of handout):

1. Find the Magnitude of \( \mathbf{v} = \langle 3, 7, 2 \rangle \)

2. Simplify the following: \( \mathbf{v} = \langle 3, 7, 2 \rangle \), \( 5\mathbf{v} = ? \)

3. \( \langle 3, 7, 2 \rangle + 4 \langle 1, 2, 3 \rangle \)

4. Write \( \mathbf{v} = \langle 3, 7, 2 \rangle \) in terms of \( \mathbf{i}, \mathbf{j}, \mathbf{k} \)

5. \( \langle 3, 7, 2 \rangle \cdot \langle 1, 2, 3 \rangle \)

6. \( \mathbf{v} = \langle 3, 7, 2 \rangle \), \( \mathbf{u} = \langle 1, 2, 3 \rangle \), \( \text{proj}_\mathbf{u} \mathbf{v} = ? \)

7. \( \mathbf{v} = \langle 3, 7, 2 \rangle \), \( \mathbf{u} = \langle 1, 2, 3 \rangle \), \( \mathbf{v} \times \mathbf{u} = ? \)
IV. Vector-Valued Functions:

A vector-valued function is essentially a 3-D parameterization where we think of the output as a vector instead of a point: \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \).

As \( t \) varies, the tail of the vector stays at the origin and the head of the vector traces out the 3-D parametric curve.

**Equation of a Vector-Valued Line:**

An equation of the line passing through the point \( P_0(x_0, y_0, z_0) \) and in the direction of the vector \( \mathbf{v} = \langle a, b, c \rangle \) is given by the vector-valued function \( \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \), or, equivalently, \( \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \) for \(-\infty < t < \infty\). The parametric equations of the line are given by \( x = x_0 + at, y = y_0 + bt, z = z_0 + ct \) for \(-\infty < t < \infty\).

Example 1: Find the vector-valued function for the line which passes through the point \((1, 2, 3)\) in the direction \((4,5,6)\).

Example 2: Find the vector-valued function for the line which passes through the points \((1, 2, 3)\) and \((4, 5, 6)\).
V. Calculus of Vector-Valued Functions:

In general, it is very difficult to say anything about vector-valued functions without calculus. Thankfully, calculus on vector-valued functions is computationally very straightforward.

Limits:

**Limits of Vector-Valued Functions:**

Let \( \mathbf{r}(t) = (x(t), y(t), z(t)) \). Then, \( \lim_{t \to a} \mathbf{r}(t) = \mathbf{L} \) if \( \lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0 \).

We say \( \mathbf{r}(t) \) approached \( \mathbf{L} \) as \( t \) approached \( a \).

Computationally, this means you can just take the limit of each component of the vector:

\[
\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle
\]

Example: Find the limit of \( \mathbf{r}(t) = \left\langle 5t, e^{3t}, t^2 + 1 \right\rangle \) as \( t \to 0 \).

Continuity:

A vector-valued function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) is continuous at \( a \) if \( \lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a) \).

This just means that \( \mathbf{r}(t) \) is continuous at \( a \) if and only if \( x(t), y(t), \) and \( z(t) \) are all continuous at \( a \).

Example: Find the values of \( t \) where the following vector-valued function is not continuous.

\[
\mathbf{r}(t) = \left\langle \frac{5}{t-3}, e^t, \tan t \right\rangle \quad 0 \leq t \leq \pi
\]
Derivatives:

We define the derivative of a vector-valued function to be:

\[ r'(t) = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} \]

Derivatives of Vector-Valued Functions:

Let \( r(t) = \langle x(t), y(t), z(t) \rangle \) where \( x(t), y(t), \) and \( z(t) \) are differentiable functions on an interval \((a, b)\).

The \( r(t) \) is differentiable on \((a, b)\) and \( r'(t) = \langle x'(t), y'(t), z'(t) \rangle \).

If \( r'(t) \neq 0 \), then \( r'(t) \) is the tangent vector to the vector-valued function at the point \((x(t), y(t), z(t))\).

If \( r(t) \) is the position of an object, then \( |r'(t)| \) gives the speed of the object at \((x(t), y(t), z(t))\).

Example 1: Find the derivative of \( r(t) = \langle t, t^2 - 4, \frac{1}{4}t^3 - 8 \rangle \).

Example 2: Find the derivative of \( R(t) = \langle t^2, t^4 - 4, \frac{1}{4}t^6 - 8 \rangle \).
Derivative Rules for Vector - Valued Functions

Let \( \mathbf{u} \) and \( \mathbf{v} \) be vector-valued functions. Let \( f \) be a scalar-valued function.

Let \( \mathbf{c} \) be a constant vector. Then:

1. \( \frac{d}{dt}(\mathbf{c}) = \mathbf{0} \) (vector)

2. \( \frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t) \) (vector)

3. \( \frac{d}{dt}(f(t)\mathbf{v}(t)) = f''(t)v(t) + f(t)v'(t) \) (vector)

4. \( \frac{d}{dt}(\mathbf{v}(f(t))) = \mathbf{v}'(f(t)) f''(t) \) (vector)

5. \( \frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \) (scalar)

6. \( \frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \) (vector)

Find the derivatives of the following vector-valued functions. Is the answer a vector or a scalar?

1. \( (5t^3 + \ln t) \mathbf{r}(3t + 11) \). Give the answer in terms of the vector \( \mathbf{r} \) and \( t \).

2. \( \mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)) \)

3. \( t \langle 4t, \ln t, 3 \rangle + 7 \langle 5, 6, 1 \rangle \)
Integrals:

An indefinite integral is just an anti-derivative. Since the derivative for vector-valued functions is just the same as taking the derivative of each component, the indefinite integral of a vector-valued function is just taking the indefinite integral of each component.

\[
\int r(t)dt = \left\{ \int x(t)dt, \int y(t)dt, \int z(t)dt \right\} + C
\]

Where \( C \) is an arbitrary constant vector.

Example: Find the indefinite integral of the following vector-valued function.

\[
\int \left( e^t \mathbf{i} + 12 \mathbf{j} + \cos(3t) \mathbf{k} \right) dt
\]

We can define a definite integral of a vector-valued function similarly.

\[
\int_{a}^{b} r(t)dt = \left\{ \int_{a}^{b} x(t)dt, \int_{a}^{b} y(t)dt, \int_{a}^{b} z(t)dt \right\}
\]

Example: Find the definite integral of the following vector-valued function.

\[
\int_{3}^{5} \left( (4 + 7t) \mathbf{i} + \mathbf{j} - \sqrt{t} \mathbf{k} \right) dt
\]
### Important Vector Operations: \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle, \quad \mathbf{v} = \langle v_1, v_2, v_3 \rangle \)

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
<th>Formula</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Magnitude</strong></td>
<td>length</td>
<td>(</td>
<td>\mathbf{v}</td>
</tr>
<tr>
<td><strong>Multiplying by a scalar</strong></td>
<td>stretching</td>
<td>( c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle )</td>
<td></td>
</tr>
<tr>
<td><strong>Adding</strong></td>
<td>Putting end to tail</td>
<td>( \mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle )</td>
<td>u+v</td>
</tr>
<tr>
<td><strong>Basis Vectors</strong></td>
<td>you can write all the others in terms of 3 main vectors</td>
<td>( \mathbf{i} = \langle 1, 0, 0 \rangle ) ( \mathbf{k} = \langle 0, 1, 0 \rangle ) ( \mathbf{l} = \langle 0, 0, 1 \rangle ) ( \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} )</td>
<td>i j k</td>
</tr>
<tr>
<td><strong>Dot Product</strong></td>
<td>Multiplying two vectors to get a scalar. Allows you to find angles between vectors.</td>
<td>( \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 ) ( =</td>
<td>\mathbf{u}</td>
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<tr>
<td><strong>Projection</strong></td>
<td>Allows you to project ( \mathbf{u} ) onto ( \mathbf{v} )</td>
<td>( \text{proj}_\mathbf{v}\mathbf{u} =</td>
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</tr>
<tr>
<td><strong>Cross Product</strong></td>
<td>Multiplying two vectors to get a vector. The magnitude of the cross product is the area of the parallelogram.</td>
<td>( \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} &amp; \mathbf{j} &amp; \mathbf{k} \ u_1 &amp; u_2 &amp; u_3 \ v_1 &amp; v_2 &amp; v_3 \end{vmatrix} ) (</td>
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