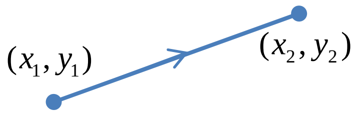
MSLC Workshop Series
Math 1172 - Workshop 4
Vector-Valued Functions

I. Parametric Curves in 2-D:

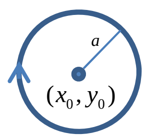
Parametric curves are curves given by $x = g(t), \quad y = h(t)$ for some independent variable *t*, usually thought of as time. So all the points on the curve can be given by $(x, y) = (g(t), h(t))$.
Caution: These curves need not be graphs of functions.

Here are some parametric curves you should be able to recognize:



A line segment from (x_1, y_1) to (x_2, y_2)

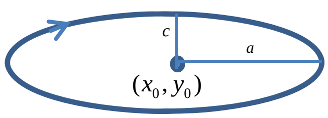
$$x = x_1 + (x_2 - x_1)t \text{ and } y = y_1 + (y_2 - y_1)t, \quad 0 \le t \le 1$$



A circle centered at (x_0, y_0) with a radius *a*.

$$x = x_0 + a\cos(bt), \quad y = y_0 + a\sin(bt)$$

The circle is generated clockwise if $b > 0$ *and counterclockwise if* $b < 0$.



An ellipse centered at (x_0, y_0):

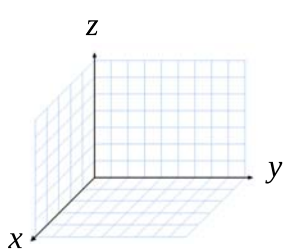
$$x = x_0 + a\cos(bt), \quad y = y_0 + c\sin(bt)$$

The circle is generated clockwise if $b > 0$ *and counterclockwise if* $b < 0$.

It is sometimes possible to **eliminate the parameter** by solving one equation for *t* and plugging it into the other equation. This will give the same curve, but you will lose the information about the direction and speed given by the parameter *t* .

You can turn functions $y = f(x)$ into parametric curves simply by letting $x = t, \quad y = f(t)$.

II. Parametric Curves in 3-D:



This is essentially exactly the same as 2-D but you get a third equation:

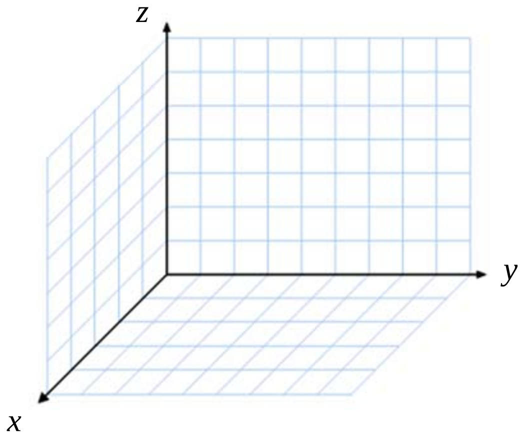
$$x = f(t), \quad y = g(t), \quad z = h(t)$$

which gives you a point in 3-space $(x, y, z) = (f(t), g(t), h(t))$

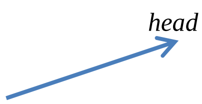
Eliminating the parameter of a 3-D curve will not give you a nice, single equation like in 2-D. For many purposes, parametric descriptions are the most natural way to describe higher dimensional curves.

Example: What does the following parametric equation look like? Describe its properties.

$$x = 3 + 2\cos t,\ y = 4 + 2\sin t,\ z = 2t \qquad 0 \le t \le 6\pi$$

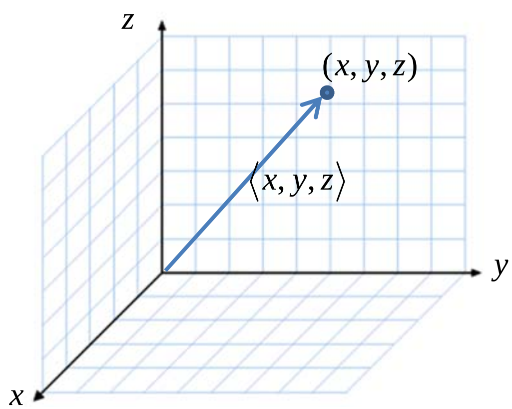


III. Vectors:



In 3-D, it is often more helpful to talk about vectors instead of points. A **vector** is an object with a **magnitude** (length) and a **direction**. We draw it as an arrow.

It does not matter where a vector is sitting in space, but if a vector $\langle x, y, z\rangle$ has its tail at the origin, then its head will be at the point (x, y, z). (People often interchange these two related but distinct concepts.)



Vector Operations Examples:

(Table on last page of handout):

1. Find the Magnitude of $\mathbf{v} = \langle 3, 7, 2 \rangle$

2. Simplify the following: $\mathbf{v} = \langle 3, 7, 2 \rangle, \quad 5\mathbf{v} = ?$

3. $\langle 3, 7, 2 \rangle + 4\langle 1, 2, 3 \rangle$

4. Write $\mathbf{v} = \langle 3, 7, 2 \rangle$ in terms of **i,j,k**

5. $\langle 3, 7, 2 \rangle \cdot \langle 1, 2, 3 \rangle$

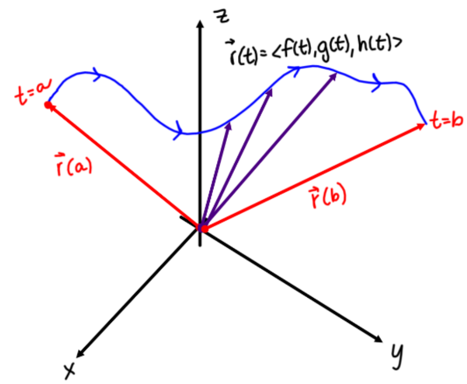
6. $\mathbf{v} = \langle 3, 7, 2 \rangle, \ \mathbf{u} = \langle 1, 2, 3 \rangle, \quad \text{proj}_{\mathbf{v}}\mathbf{u} = ?$

7. $\mathbf{v} = \langle 3, 7, 2 \rangle, \ \mathbf{u} = \langle 1, 2, 3 \rangle, \quad \mathbf{v} \times \mathbf{u} = ?$

IV. Vector-Valued Functions:

A vector-valued function is essentially a 3-D parameterization where we think of the output as a vector instead of a point: $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$.

As t varies, the tail of the vector stays at the origin and the head of the vector traces out the 3-D parametric curve.



Equation of a Vector - Valued Line:
An equation of the line passing through the point $P_0(x_0, y_0, z_0)$ and in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$ is given by the vector-valued function $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or, equivalently, $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$ for $-\infty < t < \infty$. The parametric equations of the line are given by $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$ for $-\infty < t < \infty$.

Example 1: Find the vector-valued function for the line which passes through the point $(1, 2, 3)$ in the direction $\langle 4, 5, 6 \rangle$.

Example 2: Find the vector-valued function for the line which passes through the points $(1, 2, 3)$ and $(4, 5, 6)$.

V. Calculus of Vector-Valued Functions:

In general, it is very difficult to say anything about vector-valued functions without calculus. Thankfully, calculus on vector-valued functions is computationally very straightforward.

Limits:

> **Limits of Vector - Valued Functions :**
> Let $\mathbf{r}(t) = (x(t), y(t), z(t))$. Then, $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$ if $\lim_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0.$
> We say $\mathbf{r}(t)$ approached $\mathbf{L}$ as t approached a.

Computationally, this means you can just take the limit of each component of the vector:

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} x(t),\ \lim_{t\to a} y(t),\ \lim_{t\to a} z(t) \right\rangle$$

Example: Find the limit of $\mathbf{r}(t) = \left\langle 5t, e^{3t}, t^2 + 11 \right\rangle$ as $t \to 0$

Continuity:

A vector-valued function $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$
is continuous at *a* if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

This just means that $\mathbf{r}(t)$ is continuous at *a* if and only if
$x(t),\ y(t),$ and $z(t)$ are all continuous at *a.*

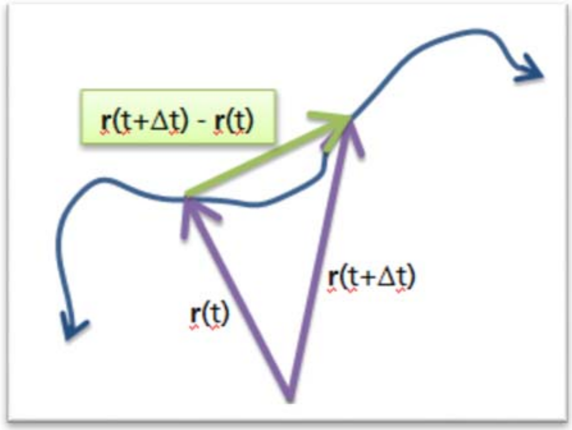
Example: Find the values of *t* where the following vector-valued function is not continuous.

$$\mathbf{r}(t) = \left\langle \frac{5}{t-3}, e^t, \tan t \right\rangle \qquad 0 \le t \le \pi$$

Derivatives:

We define the derivative of a vector-valued function to be:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$



Derivatives of Vector - Valued Functions :

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $x(t), y(t),$ and $z(t)$ are differentiable functions on an interval (a,b).
The $\mathbf{r}(t)$ is differentiable on (a,b) and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
If $\mathbf{r}'(t) \neq 0,$ then $\mathbf{r}'(t)$ is the tangent vector to the vector-valued function at the point $(x(t), y(t), z(t))$.
If $\mathbf{r}(t)$ is the position of an object, then $|\mathbf{r}'(t)|$ gives the speed of the object at $(x(t), y(t), z(t))$.

Example 1: Find the derivative of $\mathbf{r}(t) = \left\langle t, t^2 - 4, \frac{1}{4}t^3 - 8 \right\rangle$.

Example 2: Find the derivative of $\mathbf{R}(t) = \left\langle t^2, t^4 - 4, \frac{1}{4}t^6 - 8 \right\rangle$.

Derivative Rules for Vector - Valued Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be vector-valued functions. Let f be a scalar-valued function.

Let $\mathbf{c}$ be a constant vector. Then:

1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ (vector)
2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ (vector)
3. $\frac{d}{dt}(f(t)\mathbf{v}(t)) = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)$ (vector)
4. $\frac{d}{dt}(\mathbf{v}(f(t))) = \mathbf{v}'(f(t))f'(t)$ (vector)
5. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ (scalar)
6. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ (vector)

Find the derivatives of the following vector-valued functions. Is the answer a vector or a scalar?

1. $(5t^3 + \ln t)\mathbf{r}(3t + 11)$. Give the answer in terms of the vector $\mathbf{r}$ and t.

2. $\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))$

3. $t\langle 4t, \ln t, 3\rangle + 7\langle 5, 6, 1\rangle$

Integrals:

An indefinite integral is just an anti-derivative. Since the derivative for vector-valued functions is just the same as taking the derivative of each component, the indefinite integral of a vector-valued function is just taking the indefinite integral of each component.

$$\int \mathbf{r}(t)dt = \left\langle \int x(t)dt,\ \int y(t)dt,\ \int z(t)dt \right\rangle + \mathbf{C}$$

Where **C** is an arbitrary constant vector.

Example: Find the indefinite integral of the following vector-valued function.

$$\int \left(e^t \mathbf{i} + 12\mathbf{j} + \cos(3t)\mathbf{k} \right) dt$$

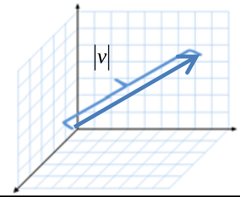
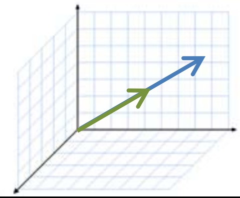
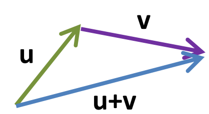
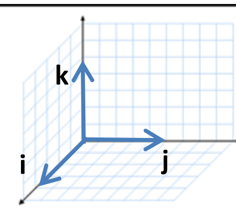
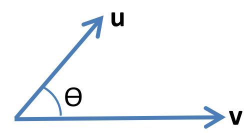
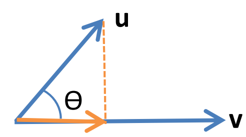
We can define a definite integral of a vector-valued function similarly.

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt,\ \int_a^b y(t)dt,\ \int_a^b z(t)dt \right\rangle$$

Example: Find the definite integral of the following vector-valued function.

$$\int_3^5 \left((4+7t)\mathbf{i} + \mathbf{j} - \sqrt{t}\mathbf{k} \right) dt$$

Important Vector Operations: $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Term	Description	Formula	Graph
Magnitude	length	$\lvert\mathbf{v}\rvert = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$	
Multiplying by a scalar	stretching	$c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle$	
Adding	Putting end to tail -or- Diagonal of the parallelogram formed by the two vectors	$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$	
Basis Vectors	you can write all the others in terms of 3 main vectors	$\mathbf{i} = \langle 1, 0, 0 \rangle$ $\mathbf{k} = \langle 0, 1, 0 \rangle$ $\mathbf{l} = \langle 0, 0, 1 \rangle$ $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$	
Dot Product	Multiplying two vectors to get a scalar. Allows you to find angles between vectors.	$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ $= \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta$	
Projection	Allows you to project **u** onto **v**	$\text{proj}_{\mathbf{v}}\mathbf{u} = \lvert\mathbf{u}\rvert\cos\theta\left(\frac{\mathbf{v}}{\lvert\mathbf{v}\rvert}\right)$ $= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$	
Cross Product	Multiplying two vectors to get a vector. The magnitude of the cross product is the area of the parallelogram.	$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ $\lvert\mathbf{u}\times\mathbf{v}\rvert = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\sin\theta$